Virtual Work – Combined Structures

4th Year

Structural Engineering

2007/8

Dr. Colin Caprani

1. Introduction

1.1 Purpose

Previously we only used virtual work to analyse structures whose members primarily behaved in flexure or in axial forces. Many real structures are comprised of a mixture of such members. Cable-stay and suspension bridges area good examples: the deck-level carries load primarily through bending whilst the cable and pylon elements carry load through axial forces mainly. A simple example is:

Our knowledge of virtual work to-date is sufficient to analyse such structures.

2. Virtual Work Overview

2.1 The Principle of Virtual Work

This states that:

> *A body is in equilibrium if, and only if, the virtual work of all forces acting on the body is zero.*

In this context, the word 'virtual' means 'having the effect of, but not the actual form of, what is specified'.

There are two ways to define virtual work, as follows.

1. Virtual Displacement:
   Virtual work is the work done by the actual forces acting on the body moving through a virtual displacement.

2. Virtual Force:
   Virtual work is the work done by a virtual force acting on the body moving through the actual displacements.

Virtual Displacements

A virtual displacement is a displacement that is only imagined to occur:

- virtual displacements must be small enough such that the force directions are maintained.
- virtual displacements within a body must be geometrically compatible with the original structure. That is, geometrical constraints (i.e. supports) and member continuity must be maintained.

Virtual Forces

A virtual force is a force imagined to be applied and is then moved through the actual deformations of the body, thus causing virtual work.

Virtual forces must form an equilibrium set of their own.

Internal and External Virtual Work

When a structures deforms, work is done both by the applied loads moving through a displacement, as well as by the increase in strain energy in the structure. Thus when virtual displacements or forces are causing virtual work, we have:

$$\begin{aligned} \delta W &= 0 \\ \delta W_I - \delta W_E &= 0 \\ \delta W_E &= \delta W_I \end{aligned}$$

where

- Virtual work is denoted δW and is zero for a body in equilibrium;
- External virtual work is δW_E, and;
- Internal virtual work is δW_I.

And so the external virtual work must equal the internal virtual work. It is in this form that the Principle of Virtual Work finds most use.

Application of Virtual Displacements

For a virtual displacement we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum F_i \cdot \delta y_i = \sum P_i \cdot \delta e_i$$

In which, for the external virtual work, F_i represents an externally applied force (or moment) and δy_i its virtual displacement. And for the internal virtual work, P_i represents the internal force (or moment) in member *i* and δe_i its virtual deformation. The summations reflect the fact that all work done must be accounted for.

Remember in the above, each the displacements must be compatible and the forces must be in equilibrium, summarized as:

Set of forces in equilibrium

$$\sum F_i \cdot \delta y_i = \sum P_i \cdot \delta e_i$$

Set of compatible displacements

Application of Virtual Forces

When virtual forces are applied, we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

And again note that we have an equilibrium set of forces and a compatible set of displacements:

Set of compatible displacements

$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i$$

Set of forces in equilibrium

In this case the displacements are the real displacements that occur when the structure is in equilibrium and the virtual forces are any set of arbitrary forces that are in equilibrium.

2.2 Virtual Work for Deflections

Deflection of a Truss

For the truss:

1. Find the forces in the members (got from virtual displacements or statics);
2. Thus we calculate the member extensions, e_i as:

$$e_i = \left(\frac{PL}{EA}\right)_i$$

3. Also, as we can choose what our virtual force δF is (usually unity), we have:

$$\begin{aligned}
\delta W &= 0 \\
\delta W_E &= \delta W_I \\
\sum y_i \cdot \delta F_i &= \sum e_i \cdot \delta P_i \\
y \cdot \delta F &= \sum \left(\frac{PL}{EA}\right)_i \cdot \delta P_i
\end{aligned}$$

4. The only remaining unknown in the virtual work equation is the actual external deflection, y. Therefore, we can calculate the deflection of the truss.

Deflections in Beams

For a beam we proceed as:

1. Write the virtual work equation for bending:

$$\begin{aligned} \delta W &= 0 \\ \delta W_E &= \delta W_I \\ y \cdot \delta F &= \sum \theta_i \cdot \delta M_i \end{aligned}$$

2. Place a unit load, δF , at the point at which deflection is required;
3. Find the real bending moment diagram, M_x, since the real rotations are given by:

$$\theta_x = \frac{M_x}{EI_x}$$

4. Solve for the virtual bending moment diagram (the virtual force equilibrium set), δM , caused by the virtual unit load.
5. Solve the virtual work equation:

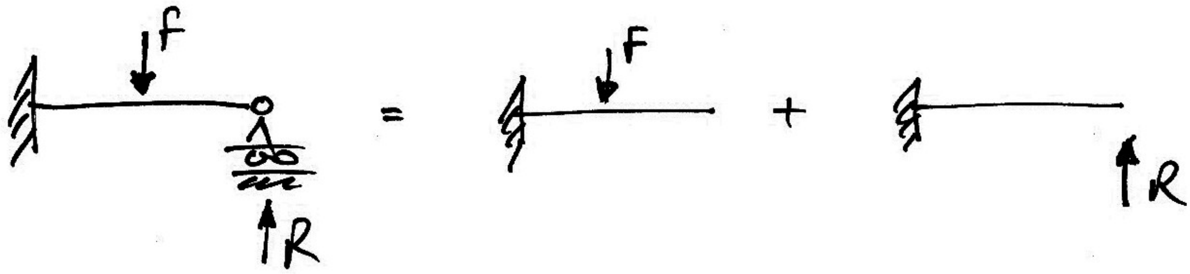
$$y \cdot 1 = \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$

6. Note that the integration tables can be used for this step.

2.3 Virtual Work for Indeterminate Structures

General Approach

Using compatibility of displacement, we have:



Final = **Primary** + **Reactant**

Next, further break up the reactant structure, using linear superposition:

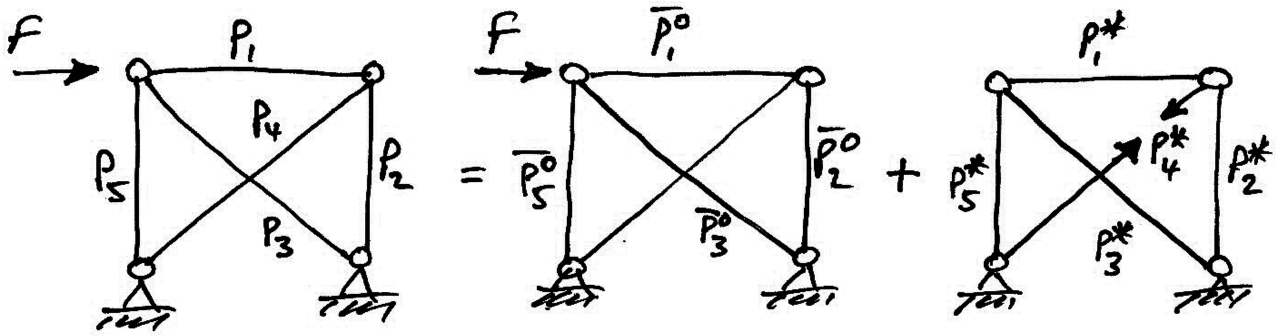
Reactant = **Multiplier** × **Unit Reactant**

We summarize this process as:

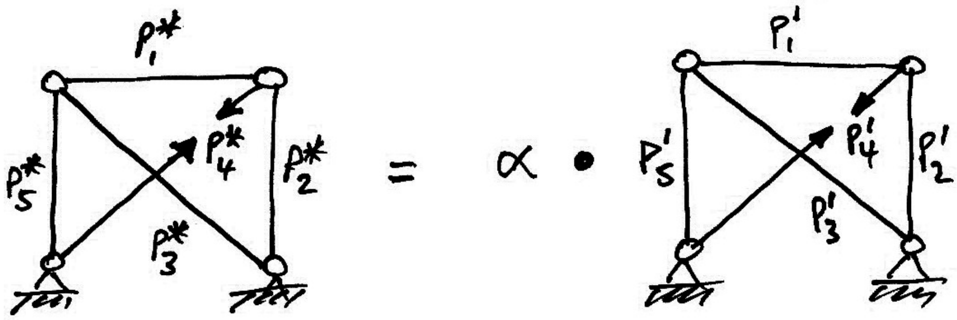
$$M = M^0 + \alpha M^1$$

- M is the force system in the original structure (in this case moments);
- M^0 is the primary structure force system;
- M^1 is the unit reactant structure force system.

For a truss, the procedure is the same:



Final = **Primary** + **Reactant**



Reactant = **Multiplier** × **Unit Reactant**

The final system forces are:

$$P = P^0 + \alpha P^1$$

The primary structure can be analysed, as can the unit reactant structure. Therefore, the only unknown is the multiplier, α.

We use virtual work to calculate the multiplier α.

Finding the Multiplier

For trusses we have:

$$
\begin{aligned}
\delta W &= 0 \\
\delta W_E &= \delta W_I \\
\sum y_i \cdot \delta F_i &= \sum e_i \cdot \delta P_i \\
0 \cdot 1 &= \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i^1 \\
0 &= \sum \left(\frac{\left(P^0 + \alpha \cdot \delta P^1 \right) L}{EA} \right)_i \cdot \delta P_i^1 \\
0 &= \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 \\
0 &= \sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \alpha \cdot \sum \frac{\left(\delta P_i^1 \right)^2 L_i}{EA_i}
\end{aligned}
$$

And for beams and frames, we have:

$$
0 = \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_i^1 \right)^2}{EI_i} dx
$$

Thus:

$$
\alpha = \frac{-\sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i}}{\sum \frac{\left(\delta P_i^1 \right)^2 L_i}{EA_i}} \text{ or } \alpha = \frac{-\sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx}{\sum \int_0^L \frac{\left(\delta M_i^1 \right)^2}{EI_i} dx}
$$

3. Virtual Work for Combined Structures

3.1 Basis

The virtual work that is done in a truss member is exactly the same concept as the virtual work done in a beam element. Thus the virtual work for a structure comprised of both types of members is just:

$$\begin{aligned} \delta W &= 0 \\ \delta W_E &= \delta W_I \\ \sum y_i \cdot \delta F_i &= \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i \end{aligned}$$

In which the first term on the RHS is the internal virtual work done by any truss members and the second term is that done by any flexural members.

If a deflection is sought:

$$\begin{aligned} y \cdot \delta F &= \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i \\ y \cdot 1 &= \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i + \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx \end{aligned}$$

To solve for an indeterminate structure, we have both:

$$M = M^0 + \alpha M^1$$

$$P = P^0 + \alpha P^1$$

For the structure as a whole. Hence we have:

$$\delta W = 0$$
$$\delta W_E = \delta W_I$$
$$\sum y_i \cdot \delta F_i = \sum e_i \cdot \delta P_i + \sum \theta_i \cdot \delta M_i$$
$$0 \cdot 1 = \sum \left(\frac{PL}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \left[\frac{M_x}{EI} \right] \cdot \delta M_x \, dx$$
$$0 = \sum \left(\frac{\left(P^0 + \alpha \cdot \delta P^1 \right) L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \left[\frac{\left(M_x^0 + \alpha M_x^1 \right)}{EI} \right] \cdot \delta M_x \, dx$$
$$0 = \sum \left(\frac{P^0 L}{EA} \right)_i \cdot \delta P_i^1 + \alpha \cdot \sum \left(\frac{\delta P^1 L}{EA} \right)_i \cdot \delta P_i^1 + \sum \int_0^L \frac{M_x^0 \cdot \delta M_x^1}{EI} dx + \alpha \cdot \sum \int_0^L \frac{\left(\delta M_x^1 \right)^2}{EI} dx$$

Hence the multiplier can be found as:

$$\alpha = - \frac{\sum \frac{P^0 \cdot \delta P_i^1 \cdot L_i}{EA_i} + \sum \int_0^L \frac{M^0 \cdot \delta M_i^1}{EI_i} dx}{\sum \frac{\left(\delta P_i^1 \right)^2 L_i}{EA_i} + \sum \int_0^L \frac{\left(\delta M_i^1 \right)^2}{EI_i} dx}$$

Note the negative sign!

Though these expressions are cumbersome, the ideas and the algebra are both simple.

Integration of Bending Moments

We are often faced with the integration of being moment diagrams when using virtual work to calculate the deflections of bending members. And as bending moment diagrams only have a limited number of shapes, a table of 'volume' integrals is used.

3.2 Examples

We will do the examples in class – keep a list of them here:

Sample Paper 2007

3. For the rigidly jointed frame shown in Fig. Q3, using Virtual Work:

(i) Determine the bending moment moments due to the loads as shown;

(*15 marks*)

(ii) Draw the bending moment diagram, showing all important values;

(*4 marks*)

(iii) Determine the reactions at *A* and *E*;

(*3 marks*)

(iv) Draw the deflected shape of the frame.

(*3 marks*)

Neglect axial effects in the flexural members.
Take the following values:
I for the frame = 150×10^6 mm^4;
Area of the stay EB = 100 mm^2;
Take E = 200 kN/mm^2 for all members.

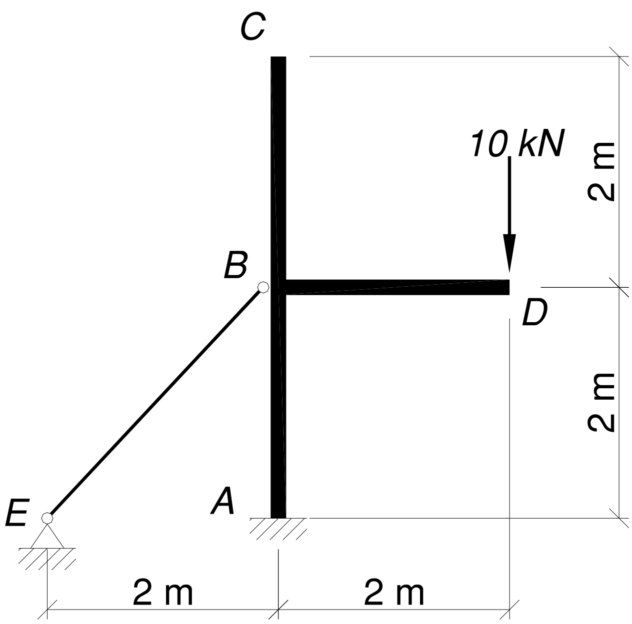


FIG. Q3

Semester 1 Exam 2007

3. For the rigidly jointed frame shown in Fig. Q3, using Virtual Work:

(i) Determine the bending moment moments due to the loads as shown;
(*15 marks*)

(ii) Draw the bending moment diagram, showing all important values;
(*4 marks*)

(iii) Determine the reactions at *A* and *E*;
(*3 marks*)

(iv) Draw the deflected shape of the frame.
(*3 marks*)

Neglect axial effects in the flexural members.
Take the following values:
I for the frame = 150×10^6 mm^4;
Area of the stay *EF* = 200 mm^2;
Take *E* = 200 kN/mm^2 for all members.

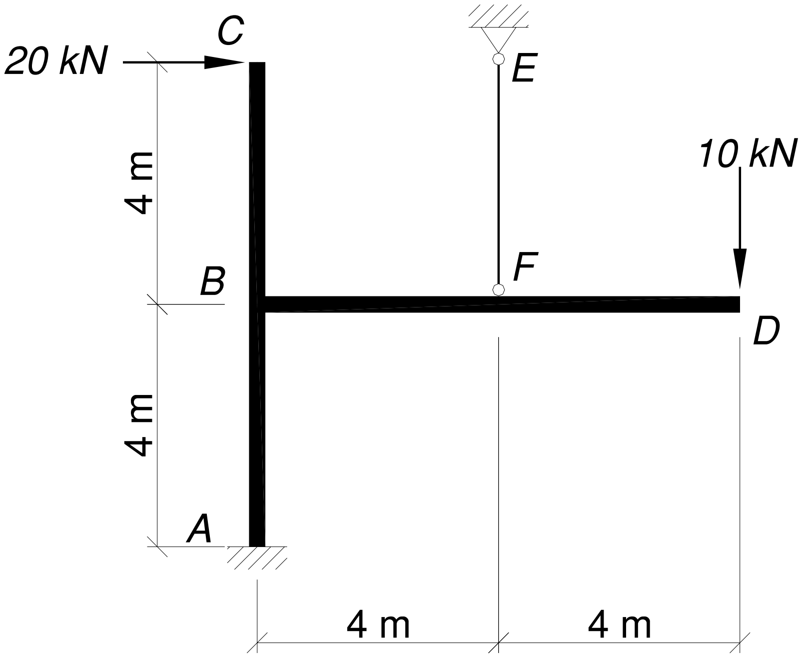
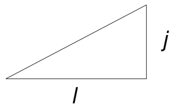
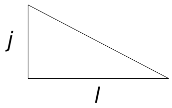
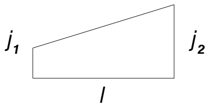
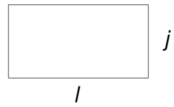
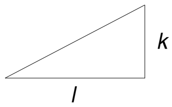
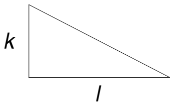
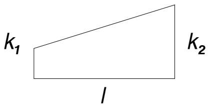
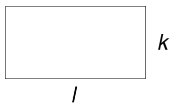
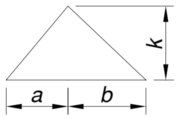
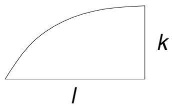
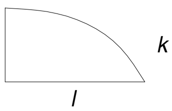
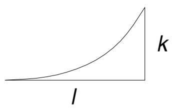
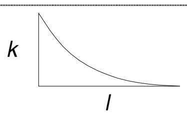
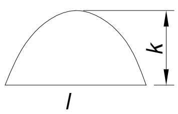


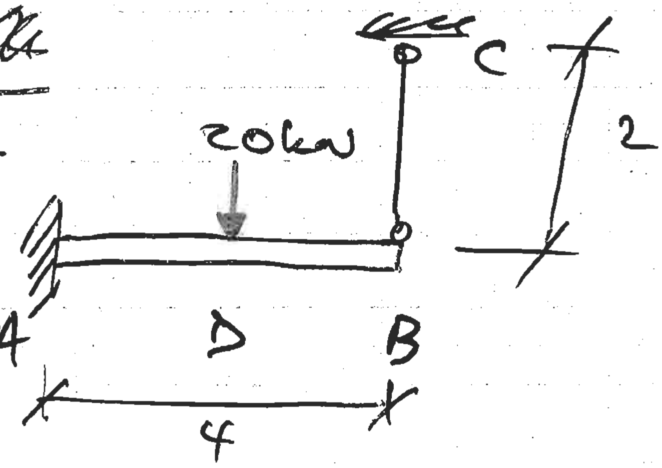
FIG. Q3

Volume Integrals

				
	$\frac{1}{3}jkl$	$\frac{1}{6}jkl$	$\frac{1}{6}\left(j_1+2j_2\right)kl$	$\frac{1}{2}jkl$
	$\frac{1}{6}jkl$	$\frac{1}{3}jkl$	$\frac{1}{6}\left(2j_1+j_2\right)kl$	$\frac{1}{2}jkl$
	$\frac{1}{6}j\left(k_1+2k_2\right)l$	$\frac{1}{6}j\left(2k_1+k_2\right)l$	$\frac{1}{6}\left[j_1\left(2k_1+k_2\right)+ j_2\left(k_1+2k_2\right)\right]l$	$\frac{1}{2}j\left(k_1+k_2\right)l$
	$\frac{1}{2}jkl$	$\frac{1}{2}jkl$	$\frac{1}{2}\left(j_1+j_2\right)kl$	jkl
	$\frac{1}{6}jk\left(l+a\right)$	$\frac{1}{6}jk\left(l+b\right)$	$\frac{1}{6}\left[j_1\left(l+b\right)+ j_2\left(l+a\right)\right]k$	$\frac{1}{2}jkl$
	$\frac{5}{12}jkl$	$\frac{1}{4}jkl$	$\frac{1}{12}\left(3j_1+5j_2\right)kl$	$\frac{2}{3}jkl$
	$\frac{1}{4}jkl$	$\frac{5}{12}jkl$	$\frac{1}{12}\left(5j_1+3j_2\right)kl$	$\frac{2}{3}jkl$
	$\frac{1}{4}jkl$	$\frac{1}{12}jkl$	$\frac{1}{12}\left(j_1+3j_2\right)kl$	$\frac{1}{3}jkl$
	$\frac{1}{12}jkl$	$\frac{1}{4}jkl$	$\frac{1}{12}\left(3j_1+j_2\right)kl$	$\frac{1}{3}jkl$
	$\frac{1}{3}jkl$	$\frac{1}{3}jkl$	$\frac{1}{3}\left(j_1+j_2\right)kl$	$\frac{2}{3}jkl$

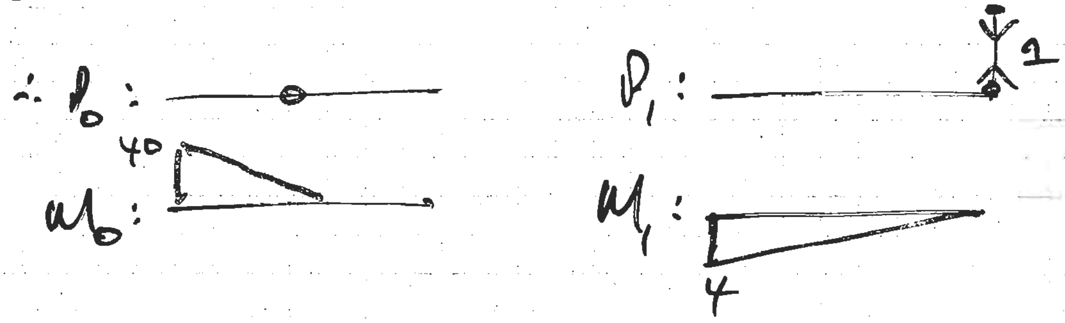
~~Problem~~

Example 1



EA & EI As per Problem 1

Choose cable as redundant.



Term ① = 0

② = $1^2 \times 2 / 16\times10^3 = 1.25\times10^{-4}$

③ : $\int_{AD} \frac{M_1 M_0 \, ds}{EI} = \frac{1}{6EI}(40)(-2 + 2(-4))2$

$= -0.0166$

$= -16.667\times10^{-3}$

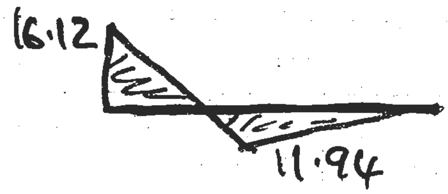
④ : $\int \frac{M_1^2 \, ds}{EI} = \frac{1}{3EI}(-4)(-4)4 = 2.667\times10^{-3}$

$\therefore \; 0 = 0 + \alpha_1(1.25\times10^{-4}) + (-16.67\times10^{-3}) + \alpha_1(2.667\times10^{-3})$

$\therefore \; \alpha_1 = 5.97$

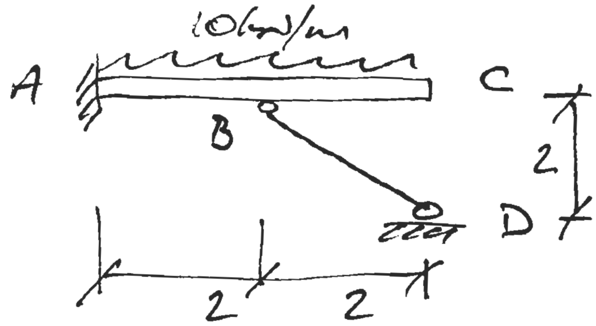
∴ Tension in cable = 5.97 kN

∴



BMD (kNm)

Example 2



$EI = 8 \times 10^3 \, kNm^2$

$EA = 16 \times 10^3 \, kN$

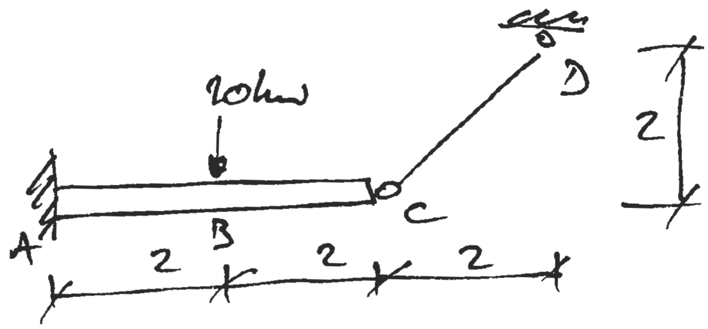
Find the BMD and δ_{CV}

For BD as redundant, $\alpha = 25.7$

$\delta_{CV} = 25mm \downarrow$

(Done in class)

Problem



$EA = 16 \times 10^3 \, kN$

$EI = 8 \times 10^3 \, kNm^2$

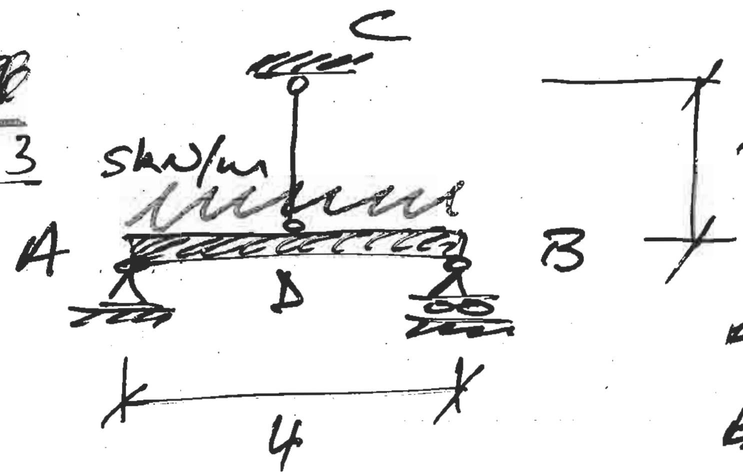
Find the BMD & δ_{cv}

For CD as redundant, $x = 7.8$

$\delta_{cv} = 1.93mm$

(Done in class)

Example 3



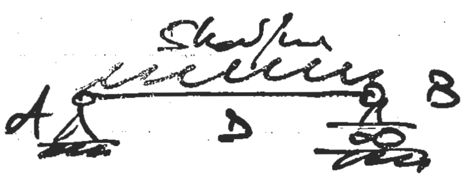
$EA = 16\times10^3\,kN$

$EI = 8\times10^3\,kNm^2$

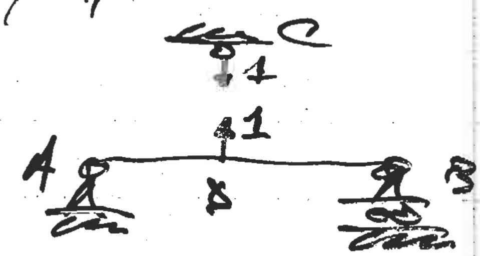
1) Solve for the BMD & force in the cable CD.
2) Determine the optimum length of the cable for efficiency of the beam.

Choose the cable as the redundant

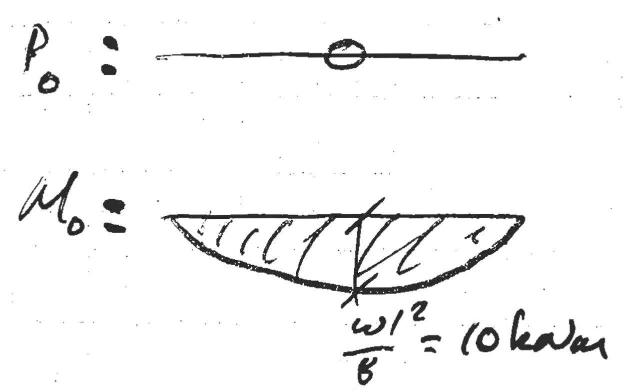
Then, our P_0/M_0 and P_1/M_1 structures are:

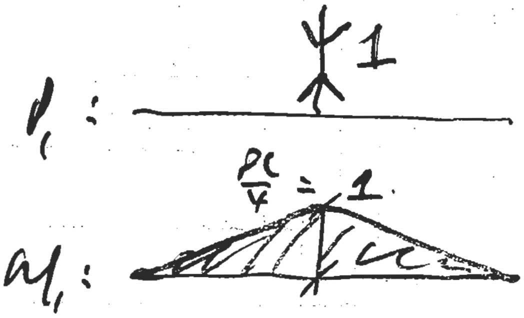


P_0/M_0



P_1/M_1





EXT. V.W. = INT. V.W.

EXT. V. FORCE × EXT. REAL DISP. = INT. V. FORCE × INT. REAL DISP

$$1 \times 0 = \Sigma P_1 p + \Sigma M_1 m$$

But, $p = PL/EA$ & $m = \dfrac{M\,ds}{EI}$

Also, $P = P_0 + \alpha P_1$ & $M = M_0 + \alpha M_1$

$$\therefore \quad 0 = \Sigma P_1 (P_0 + \alpha P_1) \frac{L}{EA} + \Sigma M_1 (M_0 + \alpha M_1) \frac{ds}{EI}$$

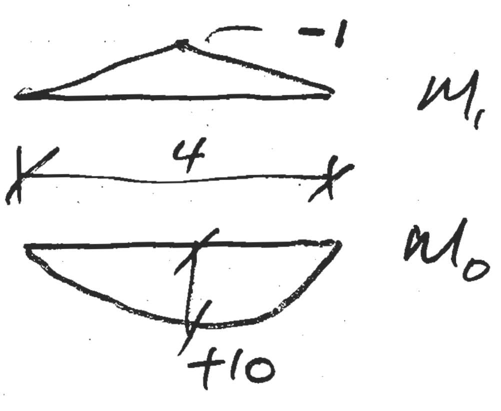
$$= \underbrace{\frac{\Sigma P_1 P_0 L}{EA}}_{1} + \alpha \underbrace{\frac{\Sigma P_1^2 L}{EA}}_{2} + \underbrace{\frac{\Sigma M_1 M_0 ds}{EI}}_{3} + \alpha \underbrace{\frac{\Sigma M_1^2 ds}{EI}}_{4}$$

Evaluate each term separately:

1: $\Sigma P_1 P_0 L / EA = 0$ as $P_0 = 0$ for all members

2: $\Sigma P_1^2 L / EA = \left(\dfrac{(1)^2 \, 2}{16 \times 10^3} \right)_{CD} = \underline{1.25 \times 10^{-4}}$

3: $\dfrac{\Sigma M_1 M_0 ds}{EI}$:



Integrate these together using the table of integration. Note there is only the following

 $= \frac{5}{12} j k l$

hence we have

$$\int_{AB} \frac{M_1 M_0 \, ds}{EI} = \frac{2}{EI} \times \left(\frac{5}{12}\right) \times (-1)(+10)(2)$$

2 halves of our BMD's

length of 1/2 shape

$$= -16.67/EI = \underline{-2.083 \times 10^{-3}}$$

(4): $\sum \frac{M_1^2 \, ds}{EI}$

i.e. $2 \times \frac{1}{3} j k l$

$$\int_{AB} \frac{M_1^2 \, ds}{EI} = \frac{2}{EI} \times \left(\frac{1}{3}\right)(-1)(-1)(2)$$

2 halves.

length of 1/2 shape

$$= \frac{1.33}{EI} = \underline{8.33 \times 10^{-5}}$$

Thus we have:

$$0 = 0 + X_1 (1.25 \times 10^{-4}) - 2.083 \times 10^{-3} + X_1 (8.33 \times 10^{-5})$$

$\Rightarrow X_1 = +10$

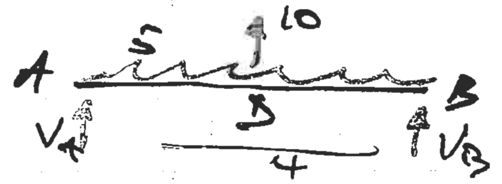
i.e. The force in the cable $= 10 \times 1 = 10 \text{ kN}$

Thus, $P = P_0 + \alpha P_1$ ⇒ cable tension = 10 kN

$M = M_0 + \alpha M_1$

⇒ At D, $M = +10 + 10(-1) = 0$

i.e.

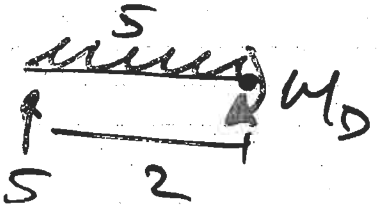


ΣM about A = 0 ∴ $5 \times 4^2/2 - 10 \times 2 - 4V_B = 0$

∴ $V_B = 5$ kN

$\Sigma F_y = 0$ ∴ $5 \times 4 - 10 - V_A - V_B = 0$

∴ $V_A = 5$ kN



ΣM about D = 0

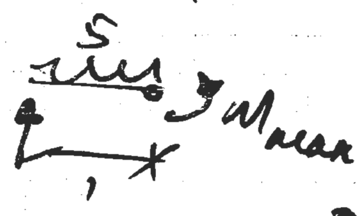
∴ $5 \times \frac{2^2}{2} - 5 \times 2 - M_D = 0$

∴ $M_D = 0$

i.e. same answer as previous

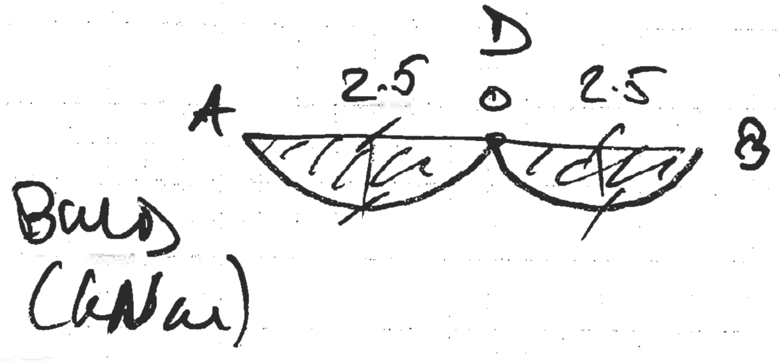
Zero shear @ 5/5 = 1m from A & B

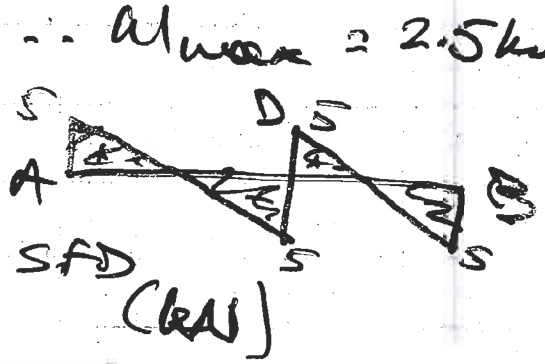
hence M_{max}:



ΣM about M_{max} = 0 ∴ $5 \times 1^2/2 - 5 + M_{max} = 0$

∴ $M_{max} = 2.5$ kNm



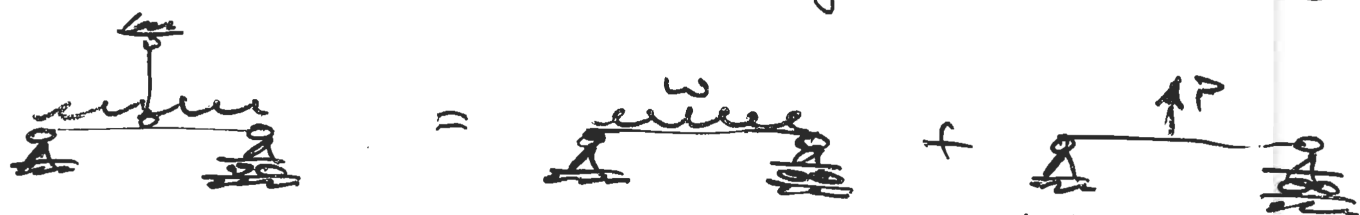


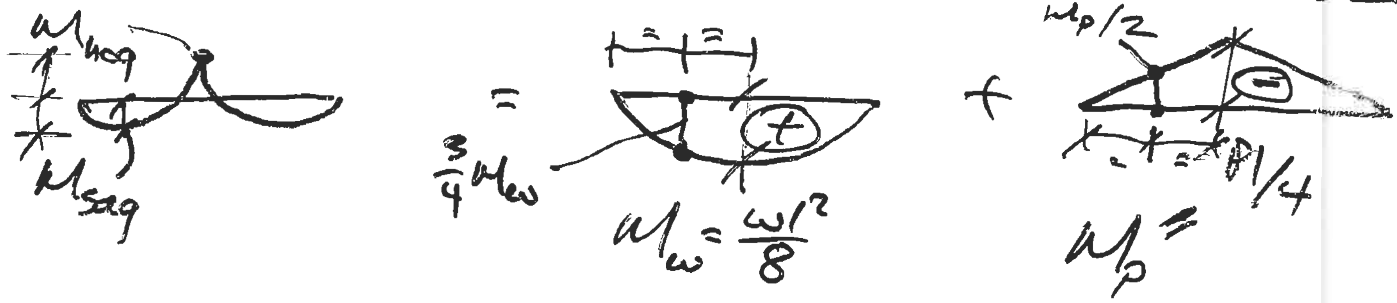
Prob 1, (2)

Efficiency of the beam means that the moments are resisted by the smallest possible beam. Then the moments (hogging & sagging) should be equal to optimise use of the material:

Qualitative BMD

The beam is the sum of two structures:





For efficiency,

$$M_{hog} = M_{sag}$$

$$\Rightarrow -M_p + M_w = +\frac{3}{4}M_w - \frac{1}{2}M_p$$

$$\Rightarrow M_w \cdot \frac{1}{4} = \frac{1}{2}M_p$$

$$\Rightarrow M_p = M_w/2$$

i.e. $PL/4 = wl^2/16 \Rightarrow P = wL/4$

There is an error here

resolved in class: $|M_{hog}| = |M_{sag}|$

we know that $w = 5$ kN/m & $C = 4$m

$\therefore P = 5$ kN

OR, $\alpha = 5.0$ from a V.W. analysis

Our virtual work equations, after taking into account the unknown length of the cable, L, becomes:

Terms 1, 3 & 4 are the same,

Term 2, $\Sigma P_1^2 L/EA = (1)^2 L/EA = \frac{L}{16\times10^3}$

$$\therefore 0 = \alpha\left(\frac{L}{16\times10^3}\right) - 2.083\times10^{-3} + \alpha\left(8.33\times10^{-5}\right)$$

$$\therefore \alpha = \frac{2.083\times10^{-3}}{\frac{L}{16\times10^3} + 8.33\times10^{-5}} = 5 \text{ — for efficiency}$$

$\therefore L = 5.33$m

OR, we could also alter EA:

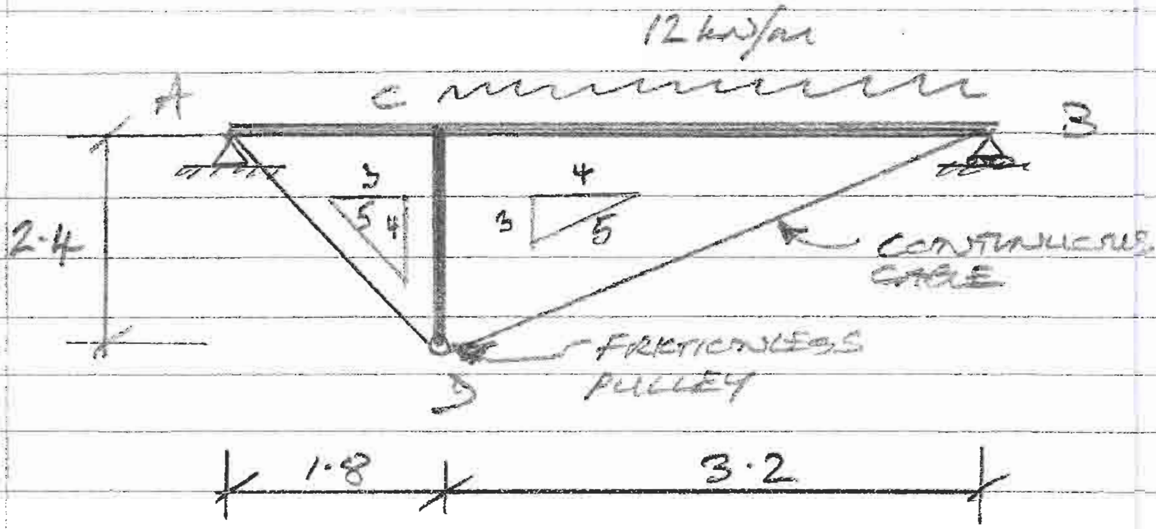
$$\text{i.e. } 5 = \frac{2.083\times10^{-3}}{\frac{2}{EA} + 8.33\times10^{-5}}$$

original length of cable in the problem.

$\therefore EA = 6\times10^3$ kN

i.e. $6/16 = 0.375$ times original EA

EXAMPLE 1



THE BEAM ACB AND STRUT CD ARE RIGIDLY CONNECTED AT C. CABLE ADB IS CONTINUOUS OVER THE FRICTIONLESS PULLEY AT D.

EI VALUES: ACB = 2EI = 8×10^3 kNm^2
CD = EI = 4×10^3 kNm^2
EA VALUES: ADB = 16×10^3 kN

AXIAL EFFECTS IN AB & CD ARE IGNORED.

THE STRUCTURE IS INDET. TO 1° IE. IF WE REMOVE THE CABLE IT IS A STAT. DET. BEAM.

=> SELECT P_{ADB} (THE CABLE FORCE) AS THE REDUNDANT FORCE.

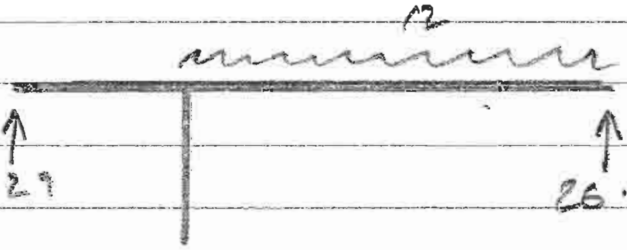
USING VIRTUAL WORK TO SOLVE, WE HAVE:

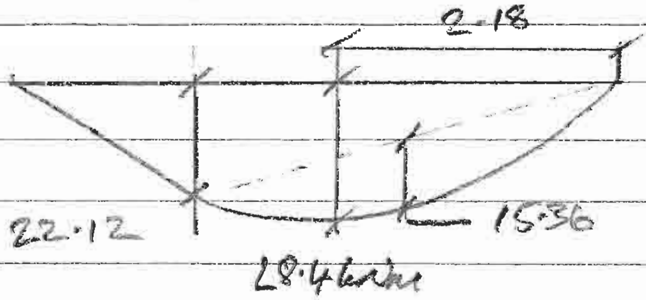
P_1 = VIRTUAL FORCE
M_1 = VIRTUAL MOMENT } DUE TO APPLIED VIRTUAL FORCE (UNIT FORCE)

ρ = REAL AXIAL DISP.
μ = REAL ROTATION

P = REAL FORCE
M = REAL MOMENT } DUE TO REAL APPLIED LOADS

P_0, M_0 ARE REAL FORCES/MOMENTS IN THE CUT-BACK STRUCTURE DUE TO THE REAL APPLIED LOADS:





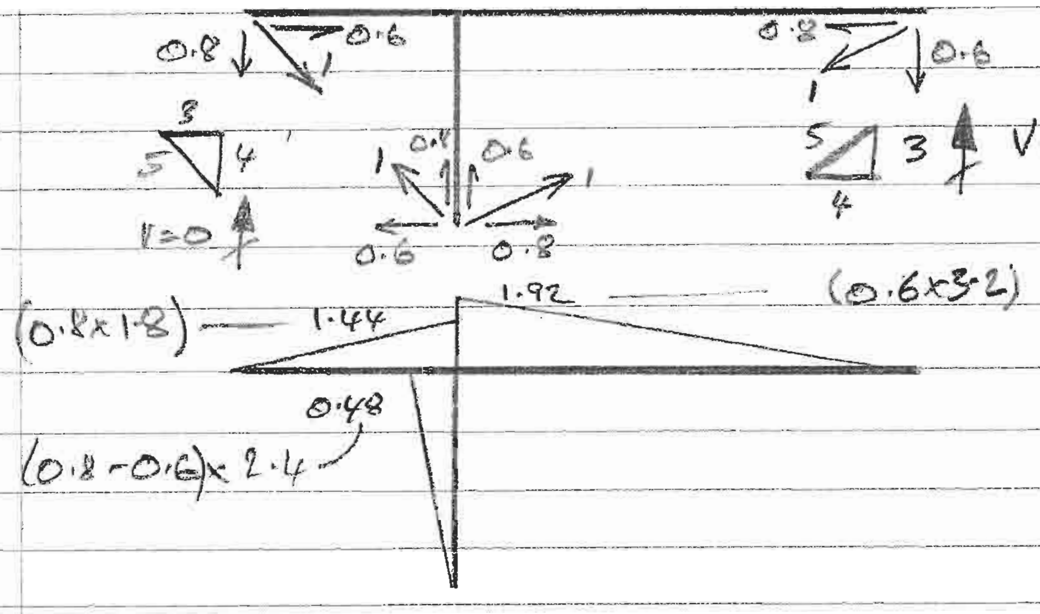
BMD.

$\underline{M_0}$

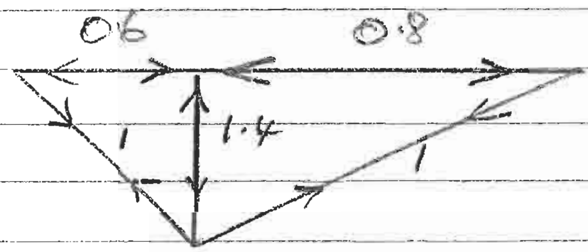
$P_0 = 0$ IN CASE ALSO.

$\underline{P_0}$

APPLY A VIRTUAL FORCE (UNIT LOAD) IN LIEU OF THE REDUNDANT:



M_1



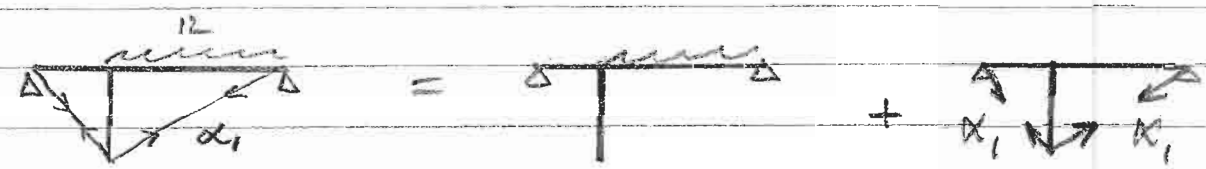
P_1

EXTERNAL VIRTUAL WORK = INTERNAL VIRTUAL WORK.

EX. VIRTUAL FORCE × EX. REAL DISPLACEMENT = INT. VIRTUAL FORCE × INT. REAL FORCE

$$1 \times 0 = \Sigma P_1 \rho + \Sigma M_1 \mu$$

ρ IS A REAL DISPLACEMENT AND μ IS A REAL ROTATION:



$$P = P_0 + \alpha_1 P_1$$
$$M = M_0 + \alpha_1 M_1$$

α_1 IS THE UNKNOWN FORCE
P_1 & M_1 ARE FORCES/MOMENTS FROM A UNIT LOAD.

HENCE A REAL DISPLACEMENT IS:

$$\rho = \frac{PL}{EA} = \frac{(P_0 + \alpha_1 P_1)L}{EA}$$

FOR A SINGLE MEMBER, SO SUM FOR ALL MEMBERS.

SIMILARLY, FROM MOHR I:

$$m = \frac{M\,ds}{EI} = (M_0 + \alpha_1 M_1)\frac{ds}{EI}$$

AGAIN, SUM FOR ALL MEMBERS. RETURNING TO THE V.W EXPRESSION WE NOW SEE

$$0 = \Sigma P_1 (P_0 + \alpha_1 P_1)\frac{L}{EA} + \Sigma M_1 (M_0 + \alpha_1 M_1)\frac{ds}{EI}$$

$$= \Sigma P_1 P_0 + \alpha_1 \Sigma \frac{P_1^2 L}{EA} + \Sigma M_1 M_0 \frac{ds}{EI} + \alpha_1 \Sigma \frac{M_1^2 ds}{EI}$$

NOW WE CALCULATE EACH OF THE TERMS IN THE ABOVE EXPRESSION:

- $\Sigma P_1 P_0$

P_0 / M_0 RELEASED STRUCTURE: $P_0 = 0$

$$\Rightarrow \Sigma P_1 P_0 = 0$$

- $\Sigma \frac{P_1^2 L}{EA}$

AS WE ARE NEGLECTING AXIAL EFFECTS IN THE BEAM MEMBERS ACB & CD, WE ONLY NEED EVALUATE THE EXPRESSION FOR MEMBERS AD, DB.

MEMBER	P_1	P_1^2	L	EA	P_1^2L/EA
AD	1	1	3	16×10^3	1.875×10^{-4}
DB	1	1	4	16×10^3	2.500×10^{-4}
					$\Sigma = 0.438\times10^{-3}$

ALTHOUGH WE COULD EASILY EVALUATE TWO MEMBERS AS:

$$\frac{\Sigma P_1^2 L}{EA} = \frac{(1)^2(3)}{16\times10^3} + \frac{(1)^2(4)}{16\times10^3} = 0.438\times10^{-3}$$

WHEN WE HAVE MANY MORE MEMBERS THE TABULAR FORM IS BETTER AS ERRORS ARE LESS LIKELY.

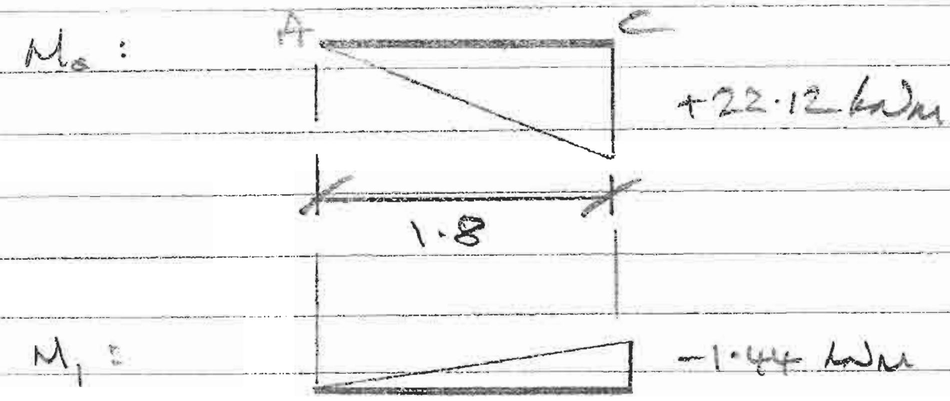
- $\dfrac{\Sigma M_1 M_0 \, ds}{EI}$

TO DETERMINE THE VIRTUAL WORK DONE BY A MOMENT, WE USE THE INTEGRAL TABLES.

IGNORE MEMBER CD AS M_0 IS ZERO IN THIS MEMBER

ALSO, SAGGING MOMENT IS POSITIVE AND A HOGGING MOMENT IS NEGATIVE.

• LENGTH AC :



$$\int M_1 M_0 \, ds = \frac{1}{3} j k l$$

j is M_0 MOMENT, k is M_1 MOMENT

$$\Rightarrow \int \left(\frac{M_1 M_0 \, ds}{EI} \right)_{AC} = \frac{1}{EI} \left[\frac{1}{3} (-1.44)(22.12)(1.8) \right]$$

$$= \frac{-19.11}{EI_{AC}}$$

$$= -2.388 \times 10^{-3}$$

• LENGTH CB :

WE WILL SPLIT THE M_0 DIAGRAM FOR MEMBER CB UP INTO CONSTITUANT PARTS THAT ARE GIVEN IN THE INTEGRAL TABLES.

$$\frac{\Sigma M_1^2 \, ds}{EI}$$

WE INTEGRATE THE M_1 DIAGRAM AGAINST ITSELF, USING THE INTEGRAL TABLES:

TRIANGLE'S : $\frac{1}{3} jkl$.

- LENGTH CD :

$$\frac{1}{EI}\left[\frac{1}{3}(0.48)(0.48)(2.4)\right] = \frac{0.184}{EI}$$

- LENGTH AC :

$$\frac{1}{EI}\left[\frac{1}{3}(-1.44)(-1.44)(1.8)\right] = \frac{1.244}{EI}$$

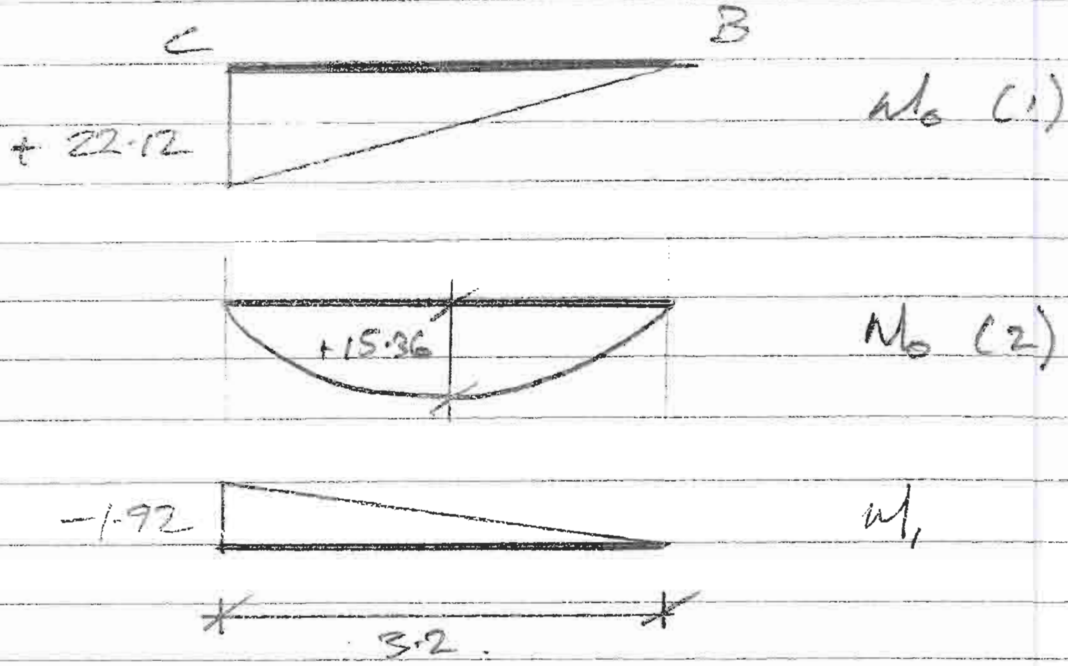
- LENGTH CB :

$$\frac{1}{EI}\left[\frac{1}{3}(-1.92)(-1.92)(3.2)\right] = \frac{3.932}{EI}$$

EI FOR ACB $= 8\times10^3$ kNm^2
EI FOR CD $= 4\times10^3$ kNm^2

$$\Rightarrow \frac{1}{8\times10^3}\left[1.244 + 3.932\right] + \frac{1}{4\times10^3}\left[0.184\right]$$

$$= 0.693\times10^{-3}$$



(1) $$\int \frac{M_1 M_0(1)\,ds}{EI} = \frac{1}{EI}\cdot\frac{1}{3}\,j\,k\,l$$

$$\Rightarrow \int \left(\frac{M_1 M_0\,ds}{EI}\right)_{CB(1)} = \frac{1}{EI}\left[\frac{1}{3}(-1.92)(22.12)(3.2)\right]$$

$$= \frac{-45.30}{EI}$$

(2) $$\int \frac{M_1 M_0(2)\,ds}{EI} = \frac{1}{EI}\cdot\frac{1}{3}\,j\,k\,l$$

$$= \frac{1}{EI}\left[\frac{1}{3}\cdot(-1.92)(15.36)(3.2)\right]$$

$$= \frac{-31.46}{EI}$$

$$\Rightarrow \int \frac{M_1 M_0\,ds}{EI} = \frac{-45.30}{EI} + \frac{-31.46}{EI} = \frac{-76.76}{EI} = -9.6\times10^{-3}$$

NOW ADD AC & CB $\Rightarrow -11.99\times10^{-3}$

NOW, RETURNING TO THE V. W.
EQN :

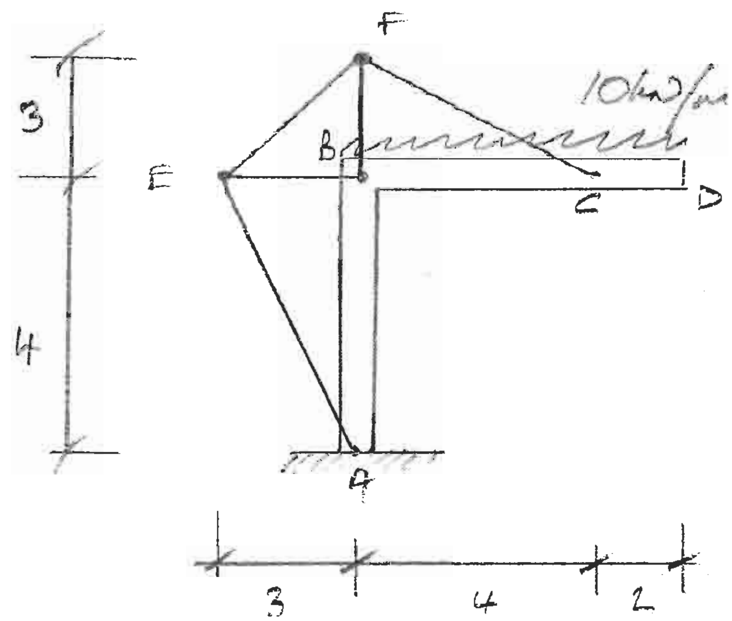
$$0 = 0 + 0.438\times10^{-3}\,\alpha_1 + (-11.99\times10^{-3}) + 0.693\times10^{-3}\,\alpha_1$$

$$\Rightarrow 1.131\,\alpha_1 = 11.99$$

$$\Rightarrow \alpha_1 = 10.60\ kN$$

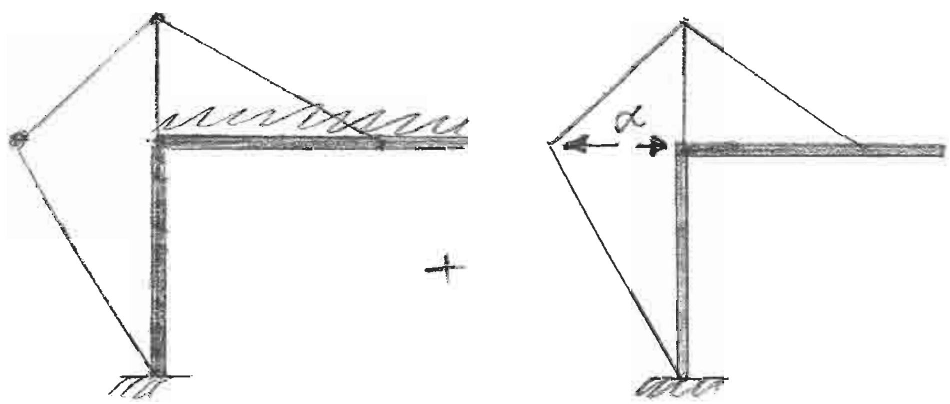
POSITIVE VALUE INDICATES THAT DIRECTION
OF α_1 CHOSEN WAS CORRECT
$\Rightarrow$ CABLE IS IN TENSION AS WE
SUSPECTED!

Q2 S'97



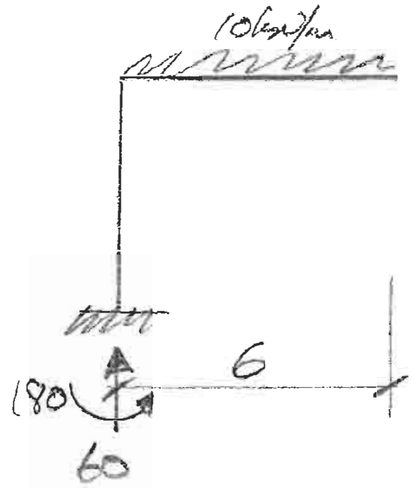
- ABCD :
  $E = 10\,kN/mm^2$
  $A = 12 \times 10^4\,mm^2$
  $I = 36 \times 10^8\,mm^4$
- AEBFC :
  $E = 200\,kN/mm^2$
  $A = 2 \times 10^3\,mm^2$

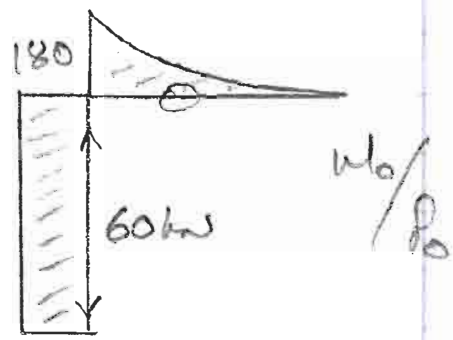
CHOOSE BE AS REDUNDANT :



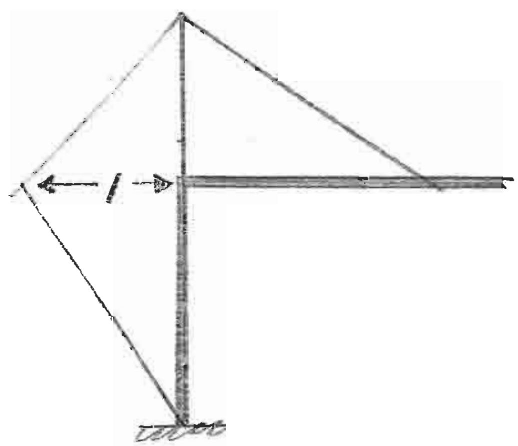
$P\,M = P_0\,M_0 + \alpha\,(P_1\ M_1)$

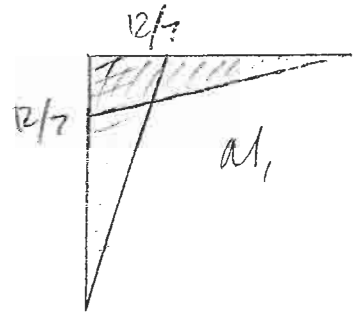
FIRST ANALYSE THE CUT BACK STRUCTURE FOR P_0, M_0.

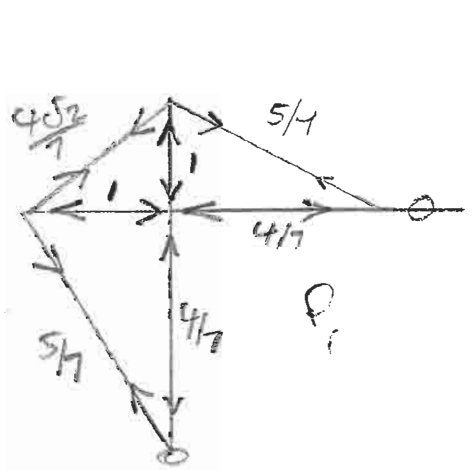


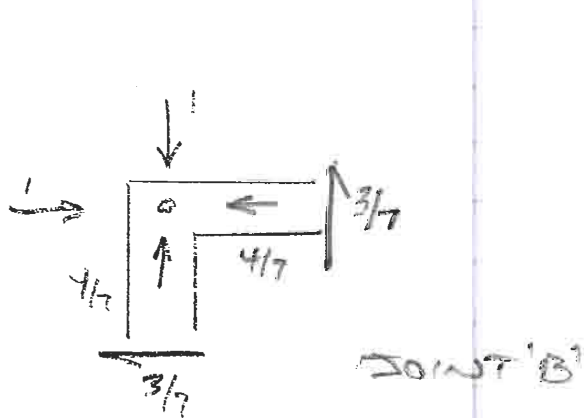


$M = 10 \times 6^2/2 = 180\,\text{kNm} \qquad V_A = 10 \times 6 = 60\,\text{kN}.$









EXT V.W = INT. V.W

EXT VIRT. FORCE $\times$ EXT. REAL DISP = INT. V. FORCE $\times$ INT. REAL DISP.

$$1 \times 0 = \Sigma \frac{P_1 P L}{AE} + \Sigma \frac{M_1 M ds}{EI}$$

$$\Rightarrow \Sigma \frac{P_1 P_0 L}{EA} + x \frac{\Sigma P_1^2 L}{EA} + \int \frac{M_1 M_0 ds}{EI} + x \int \frac{M_1^2 ds}{EI} = 0$$

$\cdot \Sigma \frac{P_1 P_0 L}{EA}$; $\Sigma \frac{P_1^2 L}{EA}$

MEMBER	P_1	P_0	EA	L	$\frac{P_1 P_0 L}{EA}$ $(\times 10^{-6})$	$\frac{P_1^2 L}{EA}$ $(\times 10^{-6})$
AB	$-4/7$	-60	1.2×10^6	4	114.3	1.088
BC	$-4/7$	0	1.2×10^6	4	0	1.088
BE	-1	0	0.4×10^6	3	0	7.5
BF	-1	0	"	3	0	7.5
AE	$5/7$	0	"	5	0	6.377
EF	$\frac{4\sqrt{2}}{7}$	0	"	$3\sqrt{2}$	0	6.927
FC	$5/7$	0	"	5	0	6.377
CD	0	0	1.2×10^6	2	0	0
				$\Sigma =$	114.3	36.86

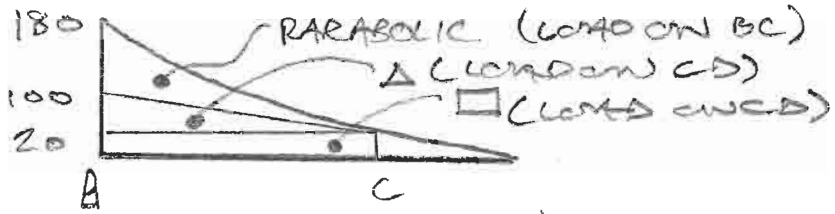
$\cdot$ ABCD : $EA = (10)(12\times10^4)$ kN $= 1.2\times10^6$ kN

$\cdot$ AEBFC : $EA = (200)(2\times10^3)$ kN $= 0.4\times10^6$ kN.

• $\int \frac{M_1 M_0 \, ds}{EI}$: $EI = (10)(36\times10^{8})\,kN\,mm^2/(10^3)^2\,(mm^2/m^2)$
$= 36\times10^{3}\,kN\,m^2$

• AB = $\frac{1}{EI}\left[\frac{1}{2}(-12/7)(180)(4)\right] = \frac{-617}{EI} = \underline{\underline{-17142.8\times10^{-6}}}$

• BC



• PARABOLIC :
$1/EI\left[\frac{1}{4}(-12/7)(80)(4)\right] = \frac{-137.14}{EI}$

• TRIANGLE :
$1/EI\left[\frac{1}{3}(-12/7)(80)(4)\right] = \frac{-182.86}{EI}$

• RECTANGLE :
$1/EI\left[\frac{1}{2}(-12/7)(20)(4)\right] = \frac{-68.57}{EI}$

$\Rightarrow$ BC $= \frac{1}{EI}(-137.14-182.86-68.57) = \underline{-10793.61\times10^{-6}}$

$\Rightarrow \int \frac{M_1 M_0 \, ds}{EI} = \underline{\underline{-27{,}936.41\times10^{-6}}}$

- $\int \frac{M_1^2 \, ds}{EI}$:

$$= \frac{2}{EI}\left[\frac{1}{3}\left(-\frac{12}{7}\right)^2(4)\right] = \underline{\underline{217.69 \times 10^{-6}}}$$

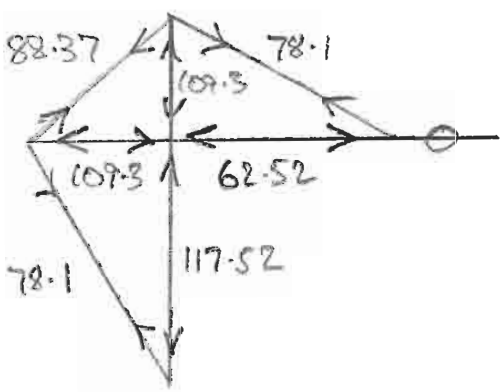
FILL VALUES INTO EQN.:

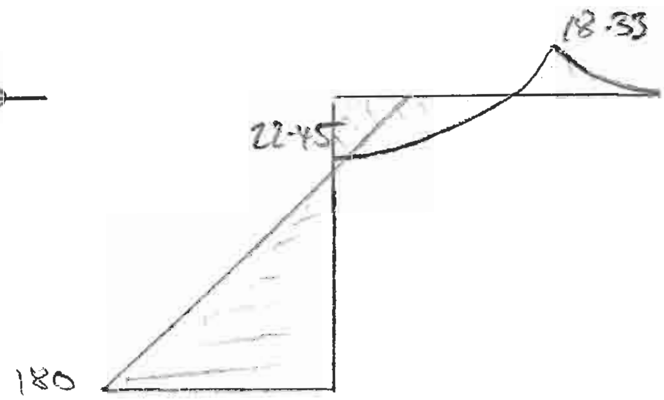
$$114.3 + 36.86\alpha - 27{,}936.41 + 217.69\alpha = 0$$

$$\Rightarrow \underline{\underline{\alpha = 109.3}}$$

FINAL VALUES:

$$P = P_0 + \alpha P_1$$
$$M = M_0 + \alpha M_1$$

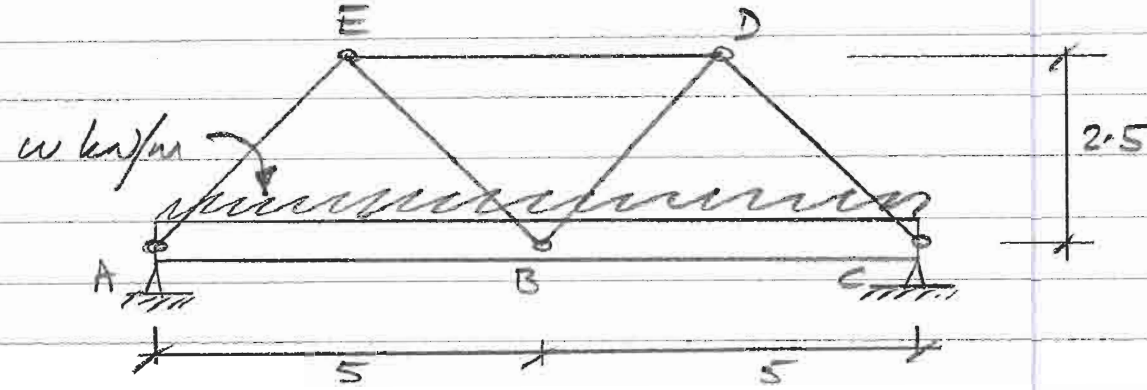




P

M

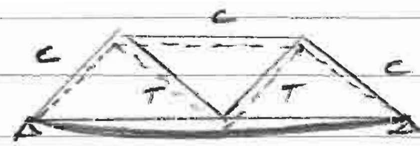
EXAMPLE 3



PROBLEM: CALCULATE THE MAX INTENSITY OF UDL, w, THAT CAN BE CARRIED BY THE BRACED SPAN ABC. THE ALLOWABLE BENDING STRESS IS $\ngtr 150\ N/mm^2$ AND THE ALLOWABLE DIRECT AXIAL STRESS (T OR C) IS $\ngtr 100\ N/mm^2$. THE BEAM ABC IS 500 mm DEEP, ITS AREA IS $6 \times 10^4\ mm^2$ AND ITS $I = 125 \times 10^7\ mm^4$. ALL OTHER MEMBERS: $A = 1000\ mm^2$. ALSO, E IS CONSTANT THROUGHOUT.

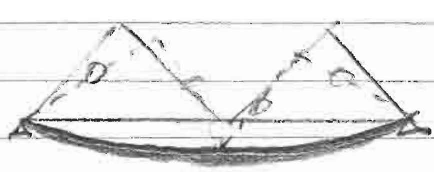
REDUNDANCY:

'IMAGINE' DSD OF THE ABOVE:



'TRUSS' ACTION

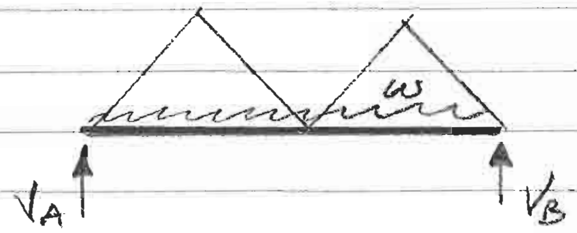
IF WE REMOVE DE:



'BEAM' ACTION.

IT IS APPARENT THAT IT IS A 1° REDUNDANT STRUCTURE. CHOOSE DE AS OUR REDUNDANT.

P_0, M_0

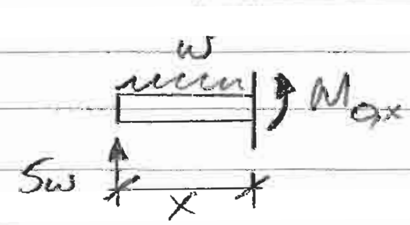


NOTE: IT IS CLEAR FROM THE PROBLEM THAT WE NEED, SOMEHOW, TO FIND AN EXPRESSION FOR THE STRESSES IN THE MEMBERS, IN TERMS OF w. ONLY THEN CAN WE PUT IN OUR MAX ALLOWABLE STRESS AND SOLVE FOR w. START BY ESTABLISHING P_0, M_0 ACTIONS IN TERMS OF w!

$\Sigma F_y = 0 \Rightarrow V_A = V_B = 5w$ (kN)

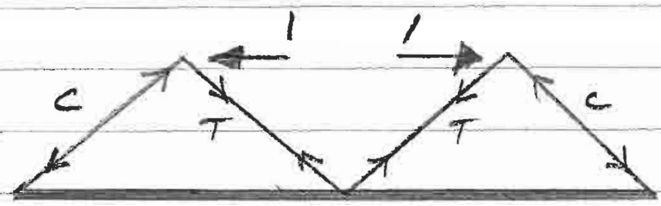
$P_0 = 0$ ALL MEMBERS.

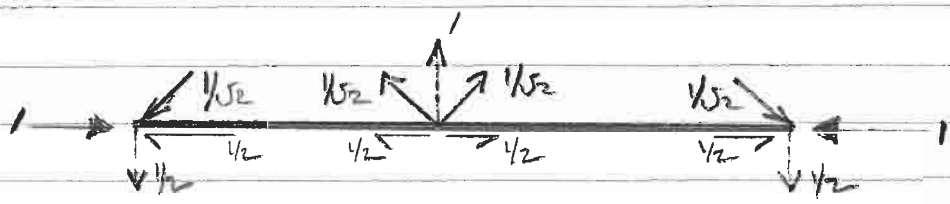
$M_0 =$

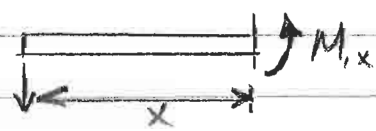


$$M_{0x} = 5wx - wx^2/2$$

P_1, M_1







NOTE: IF THE BEAM SUPPORTS WERE A PIN & ROLLER THEN WE WOULD HAVE AN AXIAL STRESS IN THE BEAM AS WELL AS THE BENDING STRESSES. HOWEVER, WE HAVE A PIN-PIN SUPPORT SYSTEM AND THE SUPPORTS THEREFORE APPLY RESTRAINT AND NO AXIAL FORCES ARE PRESENT (THE AXIAL STRESS DUE TO THE RESTRAINED LONGITUDINAL DISPLACEMENT IS CONSIDERED NEGLIGABLE). ALSO, NOTICE THERE ARE NO VERTICAL REACTIONS: $\Sigma F_y = 0$.

$$M_{1x} = -\tfrac{1}{2}x$$

- M ⊕ = TENSION IN BTM OF BEAM
- P ⊕ = AXIAL TENSION.

VIRTUAL WORK EQUATIONS:

Σ EXT. WORK $= 1 \times 0 = 0$

Σ INT. WORK $= \Sigma P_1 p + \Sigma M_1 m$

THEREFORE THE FOLLOWING APPLIES:

$$\alpha_1 = -\left[\frac{\int_0^L \frac{P_1 P_0 \, dx}{EA} + \int_0^L \frac{M_1 M_0 \, ds}{EI}}{\int_0^L \frac{P_1^2 \, dx}{EA} + \int_0^L \frac{M_1^2 \, ds}{EI}}\right]$$

IN ORDER TO EVALUATE THE VALUE OF EACH OF THE ABOVE TERMS A TABLE WOULD BE BENEFICIAL:

MEMBER	EA	EI	M_0	P_0	M_1	P_1	LIMITS
AB	6×10^{-2}	1.25×10^{-3}	$5wx - wx^2/2$	0	$-x/2$	0	$0 \to 5$
BC	6×10^{-2}	1.25×10^{-3}	$5wx - wx^2/2$	0	$-x/2$	0	$0 \to 5$
AE	1×10^{-3}	–	–	0	0	$-1/\sqrt{2}$	$0 \to 2.5\sqrt{2}$
DB	1×10^{-3}	–	–	0	0	$+1/\sqrt{2}$	$0 \to 2.5\sqrt{2}$
BE	1×10^{-3}	–	–	0	0	$+1/\sqrt{2}$	$0 \to 2.5\sqrt{2}$
DC	1×10^{-3}	–	–	0	0	$-1/\sqrt{2}$	$0 \to 2.5\sqrt{2}$
DE	1×10^{-3}	–	–	0	0	-1	$0 \to 5$

NOW WE CAN CONTINUE AND EVALUATE THE INTEGRALS.

- $\displaystyle\int_0^L \frac{P_1 P_0\, dx}{EA} = \underline{0}$ AS $P_0 = 0$ FOR ALL.

- $\displaystyle\int_0^L \frac{M_1 M_0\, dx}{EI} = 2\int_0^5 \left[(5wx - wx^2/2)(-x/2)/1.25\times10^{-3}\right] dx.$

$$= \frac{2\times10^3}{1.25}\int_0^5 \left(\frac{wx^3}{4} - \frac{5wx^2}{2}\right) dx$$

$$= \frac{2\times10^3}{1.25}\left[\frac{wx^4}{16} - \frac{5wx^3}{6}\right]_0^5$$

$$= \frac{2\times10^3 w}{1.25}\left[\frac{5^4}{16} - \frac{5\times5^3}{6}\right]$$

$$= \underline{-104,167\, w}$$

- $\displaystyle\int_0^L \frac{P_1^2\, dx}{EA}$

  - MEMBERS AE, DB, BE, DC : ◦ MEMBER DE :

$$= \frac{4}{EA}\int_0^{2.5\sqrt{2}} (\pm 1/\sqrt{2})^2 dx \quad + \frac{1}{EA}\int_0^5 (1)^2 dx$$

NOTE : EA CONSTANT, $(\pm 1/\sqrt{2})^2 = +1/2$

$$= \frac{4\times10^3}{2}\left[x\right]_0^{2.5\sqrt{2}} + 1\times10^3\left[x\right]_0^5$$

$$= 7071 + 5000 = \underline{12071}$$

$$\int_0^L \frac{M_1^2\,ds}{EI} = \frac{2\times10^3}{1{\cdot}25}\int_0^5 (-x/2)^2\,dx$$

$$= \frac{2\times10^3}{1{\cdot}25}\int_0^5 \frac{x^2}{4}\cdot dx$$

$$= \frac{2\times10^3}{1{\cdot}25}\left[x^3/12\right]_0^5$$

$$= \frac{2\times10^3}{1{\cdot}25}\cdot\frac{5^3}{12} = \underline{16{,}667}$$

<u>EVALUATE:</u>

$$\alpha_1 = -\left[\frac{-104{,}167\,w}{12{,}071 + 16{,}667}\right] = +\,3{\cdot}625\,w.$$

THE POSITIVE VALUE INDICATES THAT THE DIRECTION ASSUMED FOR THE UNIT LOAD WAS CORRECT.

<u>AXIAL LOADS</u>

$$P = P_0 + \alpha_1 P_1$$

AE & CD: $P = 0 + 3{\cdot}625w\left(-1/\sqrt{2}\right) = -2{\cdot}563\,w.$ (C)
DB & BE: $P = 0 + 3{\cdot}625w\left(+1/\sqrt{2}\right) = +2{\cdot}563\,w$ (T)
DE: $P = 0 + 3{\cdot}625\,w(-1) = -3{\cdot}625\,w$ (C)

THUS OUR IMAGINARY DSD WAS CORRECT IN ITS ASSIGNMENT OF TENSION OR COMPRESSION FORCES IN THE TRUSS.

SHEAR FORCES

WE ARE NOT REQUIRED TO ESTABLISH THESE IN THE PROBLEM.

BENDING MOMENTS

$$M = M_0 + \alpha_1 M_1$$

WE ARE ONLY CONCERNED WITH ABC:

$$M_{ABC} = \left\{ 5wx - wx^2/2 \right\} + \left\{ (3.625w)(-x/2) \right\}$$

$$= 3.1875wx - 0.5wx^2$$

Maximum Moment occurs at $\frac{dM}{dx} = 0$.

$$\Rightarrow \frac{dM}{dx} = 3.1815w - wx = 0$$

$$\Rightarrow \quad x = 3.1875\,m$$

$$\Rightarrow M_{max} = (3.1875)^2 w - \frac{(3.1875)^2 w}{2}$$

$$= 5.08w \quad (kNm)$$

DETERMINE w_{MAX}

TWO ACTIONS & ASSOCIATED STRESSES TO CHECK:

- AXIAL
- BENDING.

- AXIAL : MAX σ = 100 N/mm^2

$$\sigma_{max} = \frac{F_{max}}{A}$$

$$100 = \frac{3.625\omega \times 10^3}{1000} \quad \frac{(N)}{(mm^2)}$$

$$\Rightarrow \omega = \frac{1 \times 10^5}{3.625 \times 10^3}$$

$$= \underline{27.59 \ kN/m} \quad \left(N/mm = kN/m \right)$$

- FLEXURAL : MAX σ = 150 N/mm^2

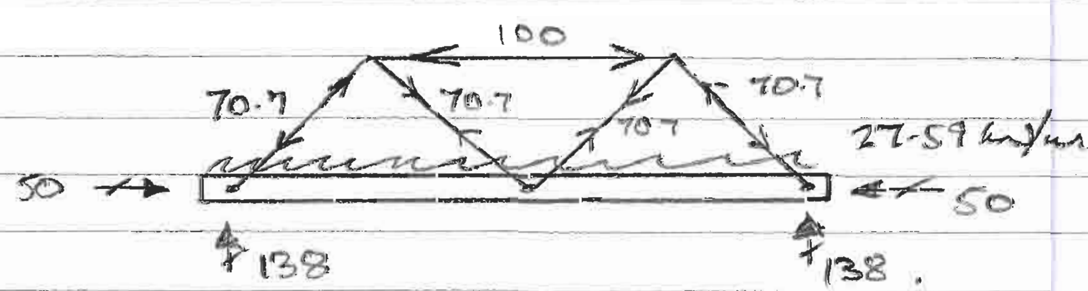
$$\sigma_{max} = \frac{My}{I}$$

$$150 = \frac{5.08\omega \times 10^6 \times (500/2)}{125 \times 10^7}$$

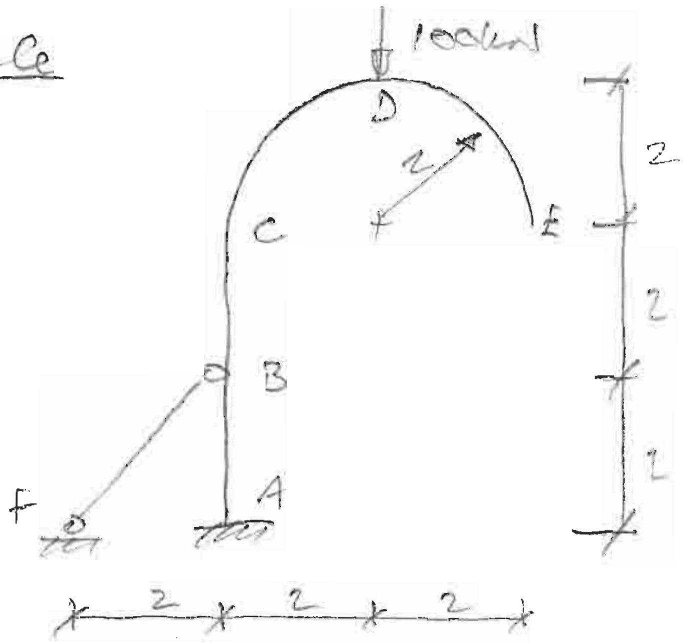
$$\Rightarrow \omega = 150 \times 0.984$$

$$= \underline{147.64 \ kN/m}$$

THUS THE MAXIMUM UDL IS DETERMINED BY THE AXIAL COMPRESSIVE FORCE IN MEMBER DE.



Example



$EI = 120 \times 10^3$ kNm²

$EA = 60 \times 10^3$ kN

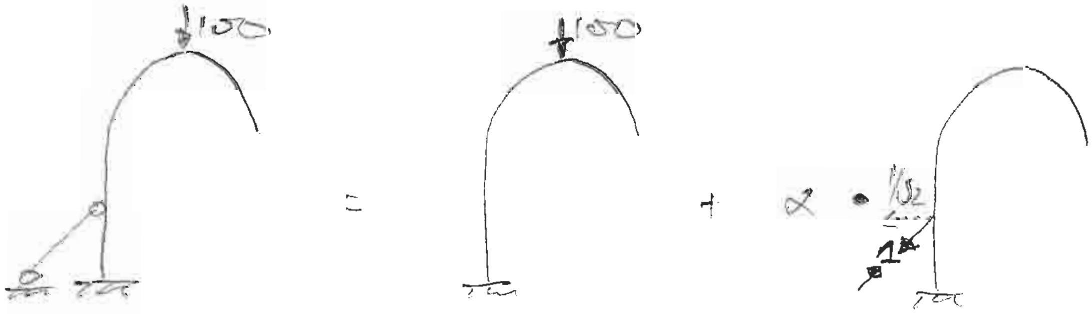
For this structure:

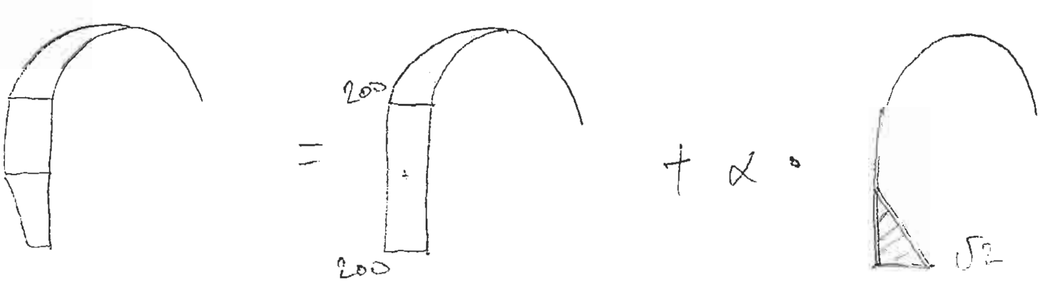
(a) draw the bending moment diagram;

(b) find δ_{Ey}

(a)

Choosing BF as the redundant gives:





$M \qquad M_0 \qquad M_1$

For virtual work:

$$\delta W = 0$$

$$\delta W_E = \delta W_I$$

$$y \cdot \delta F = \sum e \cdot \delta P + \sum \theta \cdot \delta M$$

$$0 \cdot 1 = \sum \frac{PL}{EA} \cdot \delta P + \sum \int_0^L \frac{M}{EI} \cdot \delta M \, dx$$

But $P = P_0 + \alpha P_1$ & $M = M_0 + \alpha M_1$

$$\therefore 0 = \sum \frac{(P_0 + \alpha P_1) L \delta P}{EA} + \sum \int_0^L \frac{(M_0 + \alpha M_1)}{EI} \cdot \delta M \cdot dx$$

But P_1 & δP are equivalent, as are M_1 & δM

$$\therefore 0 = \sum \frac{P_1 P_0 L}{EA} + \alpha \cdot \sum \frac{P_1^2 L}{EA} + \sum \int_0^L \frac{M_0 M_1}{EI} \cdot dx + \alpha \cdot \sum \int_0^L \frac{M_1^2}{EI} \cdot dx$$

Calculating each term separately:

- $\sum \frac{P_1 P_0 L}{EA} = 0$ since $P_0 = 0$

- $\sum \frac{P_1^2 L}{EA} = \frac{(1)^2 \cdot 2\sqrt{2}}{EA} = \frac{2\sqrt{2}}{EA}$ for member BF

- $\int_A^B \frac{M_1 M_0 \, dx}{EI} = \frac{1}{EI}\left[\frac{1}{2}(200)(-\sqrt{2})(2)\right] = \frac{-200\sqrt{2}}{EI}$

- $\int_A^B \frac{M_1^2 \, dx}{EI} = \frac{1}{EI}\left[\frac{1}{3} \cdot (-\sqrt{2})(-\sqrt{2})(2)\right] = \frac{4/3}{EI}$

Thus:

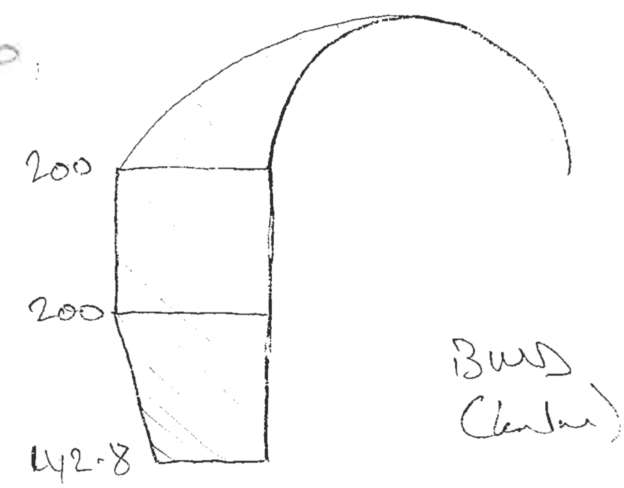
$$0 = 0 + \alpha \cdot \frac{2\sqrt{2}}{60\times10^3} - \frac{200\sqrt{2}}{120\times10^3} + \alpha \cdot \frac{4/3}{120\times10^3}$$

$$\therefore 0 = \alpha\left(4\sqrt{2} + 4/3\right) - 200\sqrt{2}$$

$$\therefore \alpha = \frac{200\sqrt{2}}{4\sqrt{2} + 4/3} = +40.46$$

Thus the force in BF is 40.46 kN in tension.

Also,



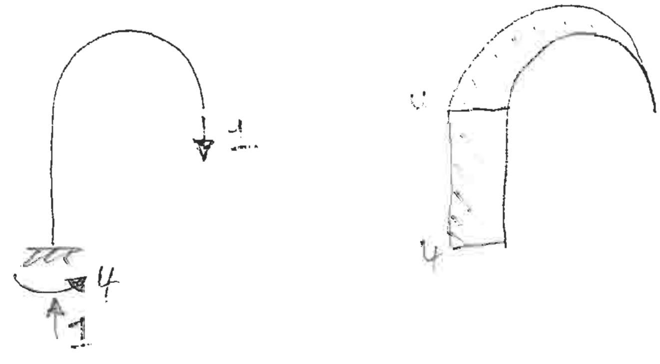
$M = M_0 + \alpha M_1$

$\therefore M_A = 200 + 40.46(-\sqrt{2})$

$= 142.8$ kNm

BMD (Closed)

(3) To find δ_{EY}, we apply a unit load at E vertically downwards. We need only use the primary structure however, since it forms an equilibrium set:

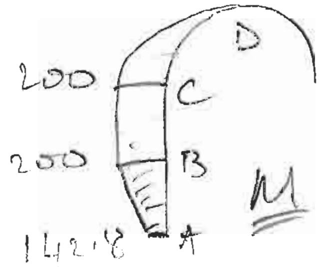


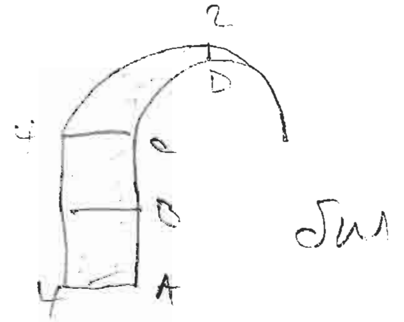
δM

For a deflection:

$$\delta_{EV}\cdot 1 = \sum \frac{PL}{EA}\cdot \delta P + \sum \int_0^L \frac{M}{EI}\cdot \delta M \cdot dx$$

Since $\delta P = 0$ we need only calculate the second term:





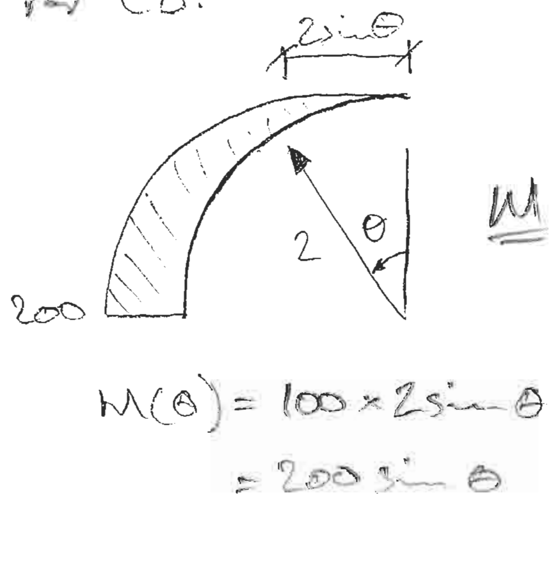
- For AB:

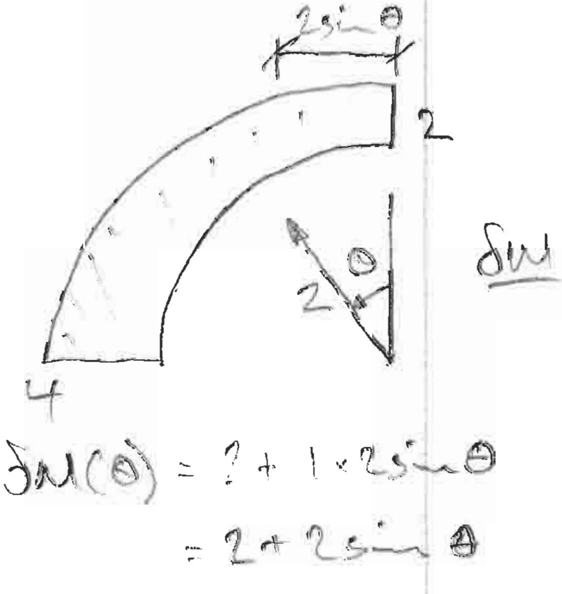
$$\int \frac{M\cdot \delta M}{EI} = \frac{1}{EI}\left[\frac{1}{2}(200+142.8)(4)(2)\right] = \frac{1371.2}{EI}$$

- For BC:

$$\int \frac{M\cdot \delta M}{EI} = \frac{1}{EI}\left[(200)(4)(2)\right] = \frac{1600}{EI}$$

- For CD:

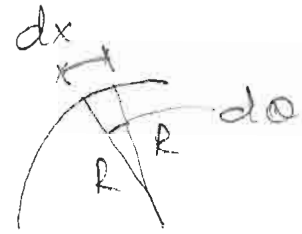




$$M(\theta) = 100 \times 2\sin\theta = 200\sin\theta$$

$$\delta M(\theta) = 2 + 1\times 2\sin\theta = 2 + 2\sin\theta$$

Also, note that since we are now integrating using the angle, we must change the dx to a $d\theta$:



$$\therefore dx = R \cdot d\theta = 2d\theta$$

Thus: $$\int_C^D \frac{M \cdot \delta m}{EI} dx = \int_0^{\pi/2} (200\sin\theta)(2+2\sin\theta) \cdot 2\,d\theta$$

$$= \int_0^{\pi/2} (800\sin\theta + 800\sin^2\theta)\,d\theta$$

$$= 800\int_0^{\pi/2} \sin\theta\, d\theta + 800\int_0^{\pi/2} \sin^2\theta \cdot d\theta$$

Taking each term in turn:

$$\int_0^{\pi/2} \sin\theta\, d\theta = -\cos\theta \Big|_0^{\pi/2} = -0-(-1) = +1$$

$$\int_0^{\pi/2} \sin^2\theta\, d\theta = \left[\frac{\theta}{2} - \frac{1}{4}\sin^2\theta\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{4} - \frac{1}{4}\cdot(1)^2\right) - \left(0 - \tfrac{1}{4}(0)^2\right)$$

$$= \pi/4 - 1/4$$

$$\therefore \int_C^D \frac{M \cdot \delta m}{EI} \cdot dx = 800(1) + 800\left(\frac{\pi}{4} - 1/4\right) = \underline{\frac{200\pi + 600}{EI}}$$

Thus:

$$\delta_{Ey} = \frac{1371.2}{EI} + \frac{1600}{EI} + \frac{200\pi + 600}{EI}$$

$$= +\frac{4199.5}{EI}$$

For $EI = 120 \times 10^3$ kNm², we have:

$$\delta_{Ey} = +35\text{mm} \downarrow$$